GENERALIZED VANDERMONDE'S SYSTEM AND LAGRANGE'S INTERPOLATION

Jean-Philippe PRÉAUX[1 2] , Jacques RAOUT[3]

Abstract. We give explicit formulas as well as a quadratic time algorithm to solve (so called *generalized Vandermonde's*) systems of linear equations with p equations and n variables. It allows in particular to find all (so called *Lagrange's interpolation*) polynoms with degree $n-1$ taking given values in p distinct given points.

Introduction

Vandermonde's linear systems of equations (Alexandre Vandermonde, french Mathematician, 1735–1796) naturally appear in numerical analysis to find Lagrange's interpolation polynom. When one wants to determine a polynom $P(x)$ with degree $n-1$ taking given values $q_1, q_2, \dots, q_n$ in n distinct points $a_1, a_2, \dots, a_n$, one has to solve the (so called *Vandermonde's*) linear system $(*)$ of equations with unknowns $w_1, w_2, \dots, w_n$:

$$\begin{cases} \omega_1 + \omega_2 a_1 + \omega_3 a_1^2 + \cdots + \omega_n a_1^{n-1} = q_1 \\ \omega_1 + \omega_2 a_2 + \omega_3 a_2^2 + \cdots + \omega_n a_2^{n-1} = q_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \omega_1 + \omega_2 a_n + \omega_3 a_n^2 + \cdots + \omega_n a_n^{n-1} = q_n \end{cases}$$

It admits a unique solution and the (so called *Lagrange's interpolation*) polynom $P(x)$ of couples $\{(a_1, q_1), (a_2, q_2), \dots, (a_n, q_n)\}$ is given by the formula :

$$P(x) = \sum_{k=1}^{n} \omega_k x^{k-1}$$

Vandermonde's linear systems also appear naturally in several problems of linear algebra.

The aim of this work is to give explicit solution to the more general interpolation problem consisting in finding all polynoms of degree $n-1$ with given values in p distinct points, where p may be different from n. It brings us to any system of p equations as above with n unknowns, that we shall extensively call *generalized Vandermonde's linear systems*.

[1]Research center of the French air force (CReA), F-13661 Salon de Provence air, France

[2]Laboratoire d'Analyse Topologie et Probabilités, Université de Provence, 39 rue F.Joliot-Curie, F-13453 marseille cedex 13, France.

E-mail : preaux@cmi.univ-mrs.fr

[3]Morpho-Analysis in Signal processing lab., Research center of the French air force (CReA), F-13661 Salon de Provence air, France.

E-mail : jraout@cr-ea.net

Mathematical subject classification. 15A06 (Primary), 65F05, 68W30, 65D05 (Secondary)

On the one hand we provide in §1 explicit formulas solving the problem in full generality, and on the other we build in §2 a quadratic time algorithm implementing the solution.

1. Explicit solutions to generalized Vandermonde's linear system

We first consider in §1.1 preliminaries notions necessary for effective computations. In §1.2 we establish explicit formulas for the inverse of a Vandermonde's square matrix, and consequently for the Lagrange's interpolation polynom. The two remaining sections are concerned with generalized Vandermonde's system : in §1.3 we explicit the kernel of a non square Vandermonde's matrix, and in §1.4 we collect all preceding result to give a solution to the general problem.

1.1. **Preliminaries : monomial coefficients.** In this section, A stands for a unitary commutative ring. Given p distinct elements $a_1, a_2, \ldots, a_p$ in A, we define for any positive integer t the *monomial coefficient of codegree t* on $a_1, a_2, \ldots, a_p$: $\sigma_{a_1 \cdots a_p}(t)$ (or more concisely $\sigma(t)$ when there is no ambiguity) informally speaking as the sum of all products of t-uples on $a_1, a_2, \ldots, a_p$ without repetition.

Definition 1. *Let $a_1, a_2, \ldots, a_p$, be p distinct elements in A. For any positive integer t, the monomial coefficient of codegree t on $a_1, a_2, \ldots, a_p$ is the element $\sigma_{a_1 \cdots a_p}(t)$ of A (or more concisely $\sigma(t)$) defined by :*

$$\begin{cases} \sigma(0) & = 1_A \\ \sigma(t) & = \underset{a_1 \cdots a_p}{\sigma}(t) = \sum_{1 \le i_1 < \cdots < i_t \le p} a_{i_1} a_{i_2} \cdots a_{i_t} \qquad \forall\, t = 1, 2, \ldots, p \\ \sigma(t) & = \underset{a_1 \cdots a_p}{\sigma}(t) = 0_A \qquad \forall\, t > p \end{cases}$$

For more convenience in transcription of formulas, we also define the related notation :

Definition 2. *Under the same hypothesis as above, and given $a = a_k$ with $1 \le k \le p$, the $\widehat{a}$-monomial coefficient of codegree t on $a_1, a_2, \ldots, a_p$, is defined as the monomial coefficient of codegree t on $a_1, \ldots, \widehat{a_k}, \ldots, a_p$, that is :*

$$\overline{\sigma}^a(t) = \underset{a_1 \cdots a_p}{\overline{\sigma}^a}(t) = \sum_{\substack{1 \le i_1 < \cdots < i_t \le p \\ i_1, \ldots, i_t \ne k}} a_{i_1} a_{i_2} \cdots a_{i_t}$$

We now explicit basic properties of monomial coefficients.

Lemma 1. *Under the same hypothesis as above, for any $t = 0, 1, \ldots, p$, one has :*

$$\sigma(t) = \overline{\sigma}^a(t) + a\, \overline{\sigma}^a(t-1)$$

Proof. Immediate by definition □

We now enonce the fundamental property of monomial coefficients which motivates them to be introduced :

Proposition 1. *In the polynom ring $A[x]$, one verifies :*

$$\prod_{i=1}^{p} (x - a_i) = \sum_{i=0}^{p} (-1)^{p-i} \underset{a_1 \cdots a_p}{\sigma}(p-i)\, x^i = \sum_{i=0}^{\infty} (-1)^{p-i} \underset{a_1 \cdots a_p}{\sigma}(p-i)\, x^i$$

Proof. Immediate by induction □

One immediately obtains by setting $x = a$:

Corollary 1. *Under the same hypothesis as above one has for any* $a = a_1, \dots, a_p$ *:*

$$\sum_{i=0}^{p} (-1)^i a^i \sigma(p-i) = 0$$

1.2. **Inversion of a square Vandermonde's matrix.** Consider the linear system $(*)$ as appearing in the introduction. Let $a_1, a_2, \dots, a_n$ be n elements of a commutative field. The *Vandermonde's square matrix* is the matrix $n \times n$ defined by :

$$V_n = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

One easily verifies that its determinant is given by :

$$\det(V_n) = \prod_{1 \le j < i \le n}^{n} (a_i - a_j) \neq 0$$

so that the matrix V_n is inversible.

Consider the family of n polynoms defined for all $j = 1, 2, \dots, n$ by :

$$P_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - a_k}{a_j - a_k} = \sum_{i=1}^{n} c_{i,j}\, x^{i-1}$$

Clearly :

$$P_j(a_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

so that the inverse matrix V_n^{-1} of V_n is given by :

$$V_n^{-1} = (c_{i,j})_{\substack{1 \le i \le n \\ 1 \le j \le n}} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix}$$

with the $c_{i,j}$ defined as above.

Let us denote by D_j the denominator of $P_j(x)$, it lies in K^*, and by $N_j(x)$ the numerator of $P_j(x)$, it lies in $K[x]$ and has degree $n-1$. It follows from proposition 1 that :

$$N_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{n} (x - a_k) = \sum_{i=1}^{n} (-1)^{i-1}\, \overline{\sigma}^{a_j}(n-i)\, x_j^{i-1}$$

$$D_j = \prod_{\substack{k=1 \\ k \neq j}}^{n} (a_j - a_k) = \sum_{i=1}^{n} (-1)^{i-1}\, \overline{\sigma}^{a_j}(n-i)\, a_j^{i-1}$$

et donc pour tout $i, j = 1, 2, \ldots, n$

$$c_{i,j} = \frac{\overline{\sigma}^{a_j}(n-i)}{\sum_{k=1}^{n}(-1)^{i+k}\ \overline{\sigma}^{a_j}(n-k)\, a_j^{k-1}}$$

Hence, the system $(*)$:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

has a unique n-uple solution $(\omega_1, \omega_2, \ldots, \omega_n)$, given by, $\forall\, i = 1, 2, \ldots, n$:

$$\omega_i = \sum_{j=1}^{n} \frac{\overline{\sigma}^{a_j}(n-i)\, q_j}{\sum_{k=1}^{n}(-1)^{i+k}\ \overline{\sigma}^{a_j}(n-k)\, a_j^{k-1}}$$

One can explicit the Lagrange's interpolation polynom with degree $n-1$ and respective values $q_1, q_2, \ldots, q_n$ in points $a_1, a_2, \ldots, a_n$:

$$P(x) = \sum_{i=1}^{n} \omega_i\, x^{i-1} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\overline{\sigma}_{a_1\cdots a_n}^{a_j}(n-i)\, q_j}{\sum_{k=1}^{n}(-1)^{i+k}\ \overline{\sigma}_{a_1\cdots a_n}^{a_j}(n-k)\, a_j^{k-1}}\ x^{i-1}$$

1.3. **Kernel of a Vandermonde's matrix $p \times n$.** In this section p and n are integers with $1 \leq p \leq n$ and $a_1, a_2, \ldots, a_p$ are distinct elements in a commutative field K. One considers the *Vandermonde's matrix with p lines and n columns* :

$$V_{p,n} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \\ 1 & a_p & a_p^2 & \cdots & a_p^{n-1} \end{pmatrix}_{p\times n}$$

The following theorem explicits a basis for the kernel of $V_{p,n}$.

Theorem 1. *The kernel of $V_{p,n}$ has dimension $n-p$ and admits a basis given by the vector $\vec{v}_1$ together with all its cyclic conjugates $\vec{v}_2, \ldots, \vec{v}_{n-p}$:*

$$\vec{v}_1 = \begin{pmatrix} (-1)^p\sigma(p) \\ \vdots \\ (-1)^i\sigma(i) \\ \vdots \\ -\sigma(1) \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \ ;\ \vec{v}_2 = \begin{pmatrix} 0 \\ (-1)^p\sigma(p) \\ \vdots \\ (-1)^i\sigma(i) \\ \vdots \\ -\sigma(1) \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ ;\cdots ;\ \vec{v}_{n-p} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ (-1)^p\sigma(p) \\ \vdots \\ (-1)^i\sigma(i) \\ \vdots \\ -\sigma(1) \\ 1 \end{pmatrix}$$

Proof. One can easily establish that $V_{p,n}$ has maximal rank. Hence, $\ker V_{p,n}$ has dimension $n-p$. It follows from corollary 1 that v_1 lies in $\ker V_{p,n}$. By multiplying equation of corollary 1 by a^k, for $0 < k < n-p$, one sees that $v_2, v_3, \ldots, v_{n-p}$ also lie in $\ker V_{p,n}$. Moreover the family of vectors $v_1, v_2, \ldots, v_{n-p}$ is clearly free, so that they constitue a basis of $\ker V_{p,n}$. □

(Note that the kernel of the Vandermonde's matrix with n lines and p columns $V_{n,p}$ has obviously dimension 0.)

1.4. **Generalized Vandermonde's system.** Collecting all the above :

Theorem 2. *The generalized Vandermonde's linear system with p equations and n unknowns :*

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \\ 1 & a_p & a_p^2 & \cdots & a_p^{n-1} \end{pmatrix}_{p,n} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_p \end{pmatrix}$$

has solutions space $\mathcal{S}$ a codimension p affine sub-space of the n-dimensional K-vector space K^n : $\mathcal{S} = \vec{\omega}_0 + \ker(V_{p,n})$, with $\vec{\omega}_0^{\,t} = (\omega_1, \omega_2, \ldots, \omega_p, 0, \ldots, 0)$, and

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_p \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{p-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{p-1} \\ \vdots & & & & \\ 1 & a_p & a_p^2 & \cdots & a_p^{p-1} \end{pmatrix}_{p,p} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_p \end{pmatrix}$$

1.5. **Example.** Let :

$$V_{n-1,n} = \begin{pmatrix} 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & & & & \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

$$\ker V_{n-1,n} = Vect(\begin{pmatrix} (-1)^{n-1}\Sigma(n-1) \\ \vdots \\ (-1)^i\Sigma(i) \\ \vdots \\ -\Sigma(1) \\ 1 \end{pmatrix}) = Vect(\begin{pmatrix} (-1)^{n-1}a_2a_3\cdots a_n \\ \vdots \\ (-1)^i\Sigma(i) \\ \vdots \\ -(a_2+a_3+\cdots+a_n) \\ 1 \end{pmatrix})$$

2. Algorithmic computations

All along the section a non negative integer p as well as p elements $a_1, a_2, \ldots, a_p$ in a unitary commutative ring A are given.

2.1. **Computation of the monomial coefficients.** The lemma 1 provides an algorithm *à la* Pascal to compute all monomial coefficients $\sigma(1), \sigma(2), \ldots, \sigma(p)$:

Proposition 2. *There is an algorithm with $O(p^2)$ complexity which computes all monomial coefficients $\sigma(t)$, for $t = 0, 1, \ldots, p$:*

Consider a matrix with $p+1$ rows and p columns, where initially the left column is constitued of 1 and on the top row is followed with zeros. Then fill in the remaining elements starting from the top row to the bottom one and from left to right using :

$$element(i, j) = element(i-1, j) + a_i.element(i-1, j-1)$$

At the end of the process the bottom row consists in $\sigma(0), \sigma(1), \ldots, \sigma(p)$.

Example. Given three elements a, b, c, we apply this method :

	0	1	2	3
	1	0	0	0
σ_a	1	a	0	0
σ_{ab}	1	a+b	ab	0
σ_{abc}	1	a+b+c	ab+ac+bc	abc

Successive computations give :

$\sigma_a(1) = 0 + a \times 1 = a$

$\sigma_{ab}(1) = \sigma_a(1) + b\,\sigma_a(0) = a + b \times 1 = a + b$
$\sigma_{ab}(2) = \sigma_a(2) + b\,\sigma_a(1) = 0 + b \times a = ab$

$\sigma_{abc}(1) = \sigma_{ab}(1) + c\,\sigma_{ab}(0) = a + b + c \times 1 = a + b + c$
$\sigma_{abc}(2) = \sigma_{ab}(2) + c\,\sigma_{ab}(1) = ab + c \times (a + b) = ab + ac + bc$
$\sigma_{abc}(3) = \sigma_{ab}(3) + c\,\sigma_{ab}(2) = 0 + c \times ab = abc$

Algorithm. Given the integer $p \geq 1$ and p distinct numbers $a_1, a_2, \ldots, a_p$, the following algorithm, written in classical algorithmic language, computes all $\sigma(t)$ for $t = 0, 1, \ldots, p$:

```
const integer          p                        // contains p
const array of number  A[p]=[a1,a2,... , ap]    // contains a1, a2, ..., ap
array of number        S[p+1]=[1,0,... ,0]
// First element of an array has index 0 !
for i=1 to p
    for j=i to 1 step -1
        S[j]=S[j]+A[i-1]*S[j-1]
// S[p+1] contains Σ(0), Σ(1), ..., Σ(p).
```

Lemma 1 also provides an algorithm to compute all coefficients $\overline{\Sigma}^a(t)$ for all $a = a_1, a_2, \ldots, a_p$ and $t = 0, 1, \ldots, p-1$.

Proposition 3. *Once all monomial coefficients $\sigma(0), \sigma(1), \ldots, \sigma(p)$, as well as $a = a_i$ are given, one can algorithmically compute the coefficients $\overline{\sigma}^a(0), \overline{\sigma}^a(1), \ldots, \overline{\sigma}^a(p-$

1) *in linear time $O(p)$, using the inductive formula :*

$$\begin{cases} \overline{\sigma}^a(0) = 1 \\ \overline{\sigma}^a(n) = \sigma(n) - a\,\overline{\sigma}^a(n-1) \end{cases}$$

En particular, using proposition 2, given p distinct elements $a_1, a_2, \ldots, a_p$, there is an algorithm which returns in quadratic time $O(p^2)$ the sequence $\sigma(1), \sigma(2), \ldots, \sigma(p)$ as well as p sequences $\overline{\sigma}^{a_i}(1), \overline{\sigma}^{a_i}(2), \ldots, \overline{\sigma}^{a_i}(p-1)$, for $i = 1, 2, \ldots, p$.

Example. Given three distinct elements a, b, c, we apply this method :

	0	1	2	3
σ	1	a+b+c	ab+ac+bc	abc
$\overline{\sigma}^a$	1	b+c	bc	0
$\overline{\sigma}^b$	1	a+c	ac	0
$\overline{\sigma}^c$	1	a+b	ab	0

Soit :

$\overline{\sigma}^a(1) = \sigma(1) - a\,\overline{\sigma}^a(0) = a + b + c - a \times 1 = b + c$
$\overline{\sigma}^a(2) = \sigma(2) - a\,\overline{\sigma}^a(1) = ab + ac + bc - a(b + c) = bc$
$\overline{\sigma}^b(1) = \sigma(1) - b\,\overline{\sigma}^b(0) = a + b + c - b \times 1 = a + c$
$\overline{\sigma}^b(2) = \sigma(2) - b\,\overline{\sigma}^b(1) = ab + ac + bc - b(a + c) = ac$
$\overline{\sigma}^c(1) = \sigma(1) - c\,\overline{\sigma}^c(0) = a + b + c - c \times 1 = a + b$
$\overline{\sigma}^c(2) = \sigma(2) - c\,\overline{\sigma}^c(1) = ab + ac + bc - c(a + b) = ab$

Algorithm. Given the integer $p \geq 1$, p numbers $a_1, a_2, \ldots, a_p$ and the sequence of monomial coefficients $\sigma(0), \sigma(1), \ldots, \sigma(p)$, the following algorithm written in standard algorithmic language returns all $\widehat{a}$-monomial coefficients $\overline{\sigma}^a(t)$ for $t = 0, 1, \ldots, p-1$ and $a = a_1, a_2, \ldots, a_p$.

```
const  integer          p                    // contains p
const  array of number  A[p]=[a1,...,ap]     // contains a1, a2, ..., ap
const  array of number  S[p+1]=[s0,s1,...,sp] // σ(0), σ(1), ..., σ(p)
array of array of number T[p][p]
for i=0 to p-1
   T[i][0]=1
   for j=1 to p-1
      T[i][j]= S[j]-a*T[i][j-1]
\\ T[i]  contains σ̄^{a_i}(0), σ̄^{a_i}(1), ..., σ̄^{a_i}(p-1)
```

References

[L] H.Lebesgue, *L'oeuvre mathématique de Vandermonde*, Enseignement Math. (2) 1 (1956), pp 203-223.